• $\displaystyle\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\frac{2}{3}$ の証明

$\displaystyle\frac{n^3-1}{n^3+1}=\frac{(n-1)n}{(n-1)^2+(n-1)+1}\times\frac{n^2+n+1}{n(n+1)}$ となり、$a_n=\displaystyle\frac{n^2+n+1}{n(n+1)}$ とおくと

$$\begin{aligned}\prod_{n=2}^{m}\frac{n^3-1}{n^3+1}&=\prod_{n=2}^{m}\frac{a_n}{a_{n-1}}=\frac{a_2}{a_1}\times\frac{a_3}{a_2}\times\frac{a_4}{a_3}\times\cdots\times\frac{a_m}{a_{m-1}}=\frac{a_m}{a_1}\\&=\frac{2(m^2+m+1)}{3(m^2+m)}=\frac{2(1+1/m+1/m^2)}{3(1+1/m)}\to\frac{2}{3}\quad(\text{as } m\to\infty).\end{aligned}$$

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